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Synchronization of an ensemble of oscillators regulated by their spatial movement

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Synchronization for a collection of oscillators residing in a finite two dimensional plane is explored. The coupling between any two oscillators in this array is unidirectional, viz., master-slave configuration. Initially the oscillators are distributed randomly in space and their autonomous time-periods follow a Gaussian distribution. The duty cycles of these oscillators, which work under an on-off scenario, are normally distributed as well. It is realized that random hopping of oscillators is a necessary condition for observing global synchronization in this ensemble of oscillators. Global synchronization in the context of the present work is defined as the state in which all the oscillators are rendered identical. Furthermore, there exists an optimal amplitude of random hopping for which the attainment of this global synchronization is the fastest. The present work is deemed to be of relevance to the synchronization phenomena exhibited by pulse coupled oscillators such as a collection of fireflies. © *2010 American Institute of Physics*. [doi:10.1063/1.3496399]

Synchronization is a natural occurrence in an ensemble of coupled oscillators. It plays a crucial role in the collective dynamics exhibited by populations of such oscillators. The abundance of these populations in nature lends the synchronous phenomena a flavor of ubiquitousness. In particular, the collective dynamics of biological oscillators, such as flashing of fireflies and swarming of fishes and bees, provide exquisite examples of the synchronization phenomena. The influence of spatial movement of such oscillators on the observed collective dynamics is addressed in the present work. In particular, our results seem to be of relevance to the technologically important field of swarm robotics.

I. INTRODUCTION

Synchronization is a universal phenomena. Starting from biological systems like swarms of bees, groups of fishes, flocks of birds to astronomical systems like solar system and other galaxies, all exhibit interesting examples of synchronization.[1–9] Although etymologically "synchronization" means events which occur at the same time, synchronization in space is also abundant. Pattern formation, cluster formation, and self-organization of different systems are manifestations of the synchronization phenomena in space. Coupling plays an important, although not necessary, role in synchronizing systems. A system undergoing synchronization may have unidirectional coupling or bidirectional coupling or no coupling at all between its constituents. The system without any inherent coupling between its constituting elements can exhibit synchronization if a common forcing is imposed on all these elements. This kind of synchronization is labeled as generalized synchronization.

In certain situations the coupling between elements of a system is mediated by emitted pulses and/or their respective phases. Two common examples for such systems are fireflies flashing in rhythm and people clapping in unison after a concert. There have been some previous attempts at theorizing the synchronization of such oscillators.[10–17] The pioneering work in this field was carried out by Strogatz and Mirollo. They took a collection of globally coupled pulsating oscillators with identical frequencies and a random phase distribution. The emitted pulse, in their work, follows an "integrate and fire" form, i.e., a curve which is concave downward and monotonically increasing with the phase of the oscillator. Upon reaching its maximum value it resets itself to the lower bound, hence restarting the cycle again. To state mathematically,

$$x = f(\phi): x, \phi \in [0,1]; f' > 0, f'' < 0, \quad (1)$$

$$\frac{d\phi}{dt} = \frac{1}{\tau}, \quad (2)$$

where x is a normalized quantity which characterizes the relative amount of "charge" of an oscillator with respect to its maximum, ϕ is its phase, and τ is the intrinsic time-period of the oscillators, a constant, for the entire ensemble. A firing event occurs whenever x attains the unit value. At each firing event other oscillators are excited by a fixed amount or brought to the firing threshold, whichever is less (i.e., x_{other} becomes $\min\{x_{\text{other}}+\epsilon, 1\}$). If a firing triggers another firing, then an absorption event occurs. It was shown by Strogatz and Mirollo that under this prescription, global synchronization is attained for almost all initial configurations. However, in the above analysis spatial movement for the oscillators and a mismatch in their autonomous frequencies are precluded, inclusion of which would make the theoretical framework more realistic.

In the present work a collection of oscillators is taken which are unidirectionally coupled and the extent of coupling is neither global nor local, rather an intermediate one.

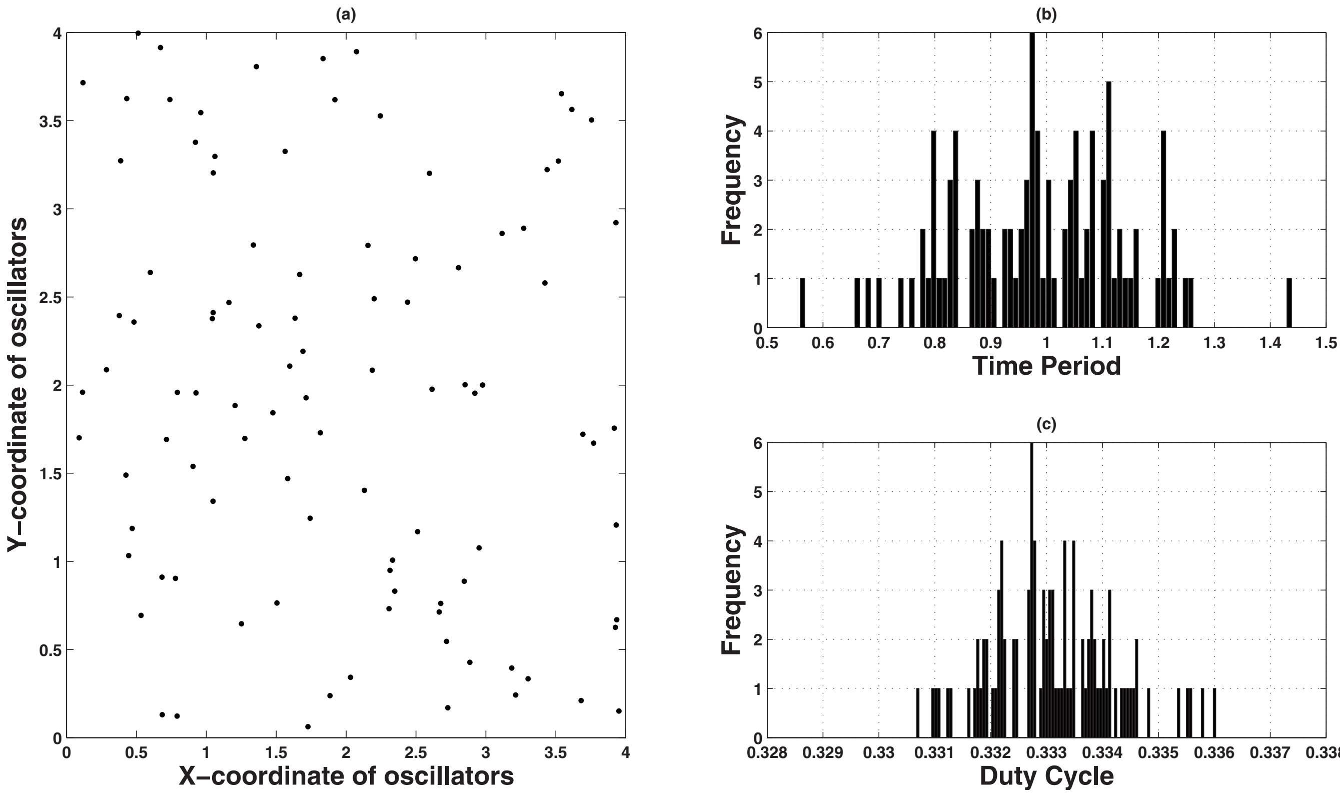

FIG. 1. One particular initial distribution of 100 oscillators. (a) Initial spatial distribution. (b) Initial time-period distribution: Gaussian (1, 0.15). (c) Initial duty cycle distribution: Gaussian (0.333, 0.001).

These oscillators synchronize through a phase difference minimization scheme. The model dynamics are computed for three distinct scenarios: (a) oscillators are static, (b) oscillators are harmonically hopping, and finally, (c) randomly hopping oscillators. Upon the implementation of a synchronization protocol (described later) to the three cases, the network of oscillators shows diverse behavior such as partial synchronization, cluster formation, and global synchronization. This global synchronization is observed only for randomly hopping oscillators. Moreover, there exists an optimum amplitude for random hopping which yields the swiftest attainment of the globally synchronized state.

II. THE MODEL

Numerical computations using MATLAB were carried out to extract the salient features of the collective dynamics for the spatiotemporal distribution of oscillators.

A. Initial configuration

The oscillator has been defined as an abstract quantity which possesses a time-period, a duty cycle, and an associated current (instantaneous) phase. Hence, the state space of a particular oscillator is defined as the triplet of time-period, duty cycle, and the phase. Therefore, the whole state space of N oscillators is defined as

$$\Re^{3N} = \{(\tau_1, d_1, \phi_1), (\tau_2, d_2, \phi_2), \ldots, (\tau_N, d_N, \phi_N)\}, \quad (3)$$

where (τ_i, d_i, ϕ_i) denotes the time-period, duty cycle, and current phase of the ith oscillator. A dimensionless $m \times m$ square plane is chosen upon which N oscillators are distributed randomly. In the present case $m=4$ and $N=100$. This choice of parameters (m, N) is completely arbitrary. The time-periods for the oscillators are normally distributed with a mean of 12 and a standard deviation of 0.15. Once again, these two numbers are dimensionless and are selected arbitrarily. The oscillators are pulse coupled and the duty cycle of a typical pulse is normally distributed with a mean of 0.333 and a standard deviation of 0.001 (dimensionless and arbitrary). Also the initial phases of the oscillators are distributed uniformly and randomly between 0 and 2π. A typical initial configuration is presented in Fig. 1. It shows the spatial distribution of the oscillators in conjunction with the normal distributions for the time-periods and duty cycles. In contrast to the analysis by Strogatz and Mirollo,[10] we chose square pulses for the present work. The actual output as seen in Strogatz–Mirollo model is a spike, which can be approximated using a square pulse with very low duty cycle. A typical pulse profile is shown in Fig. 2(a).

B. Synchronization protocol

1. Definition of an iteration and dynamics of the current phase

It is assumed that the system is being timed with a clock in the laboratory frame which has a time-period of 1. At each tick of the clock we take a snapshot of the system and perform the required calculations and update the system parameters. In other words after each 1 unit of time we repeat the process. The current phase (ϕ) is assumed to be varying

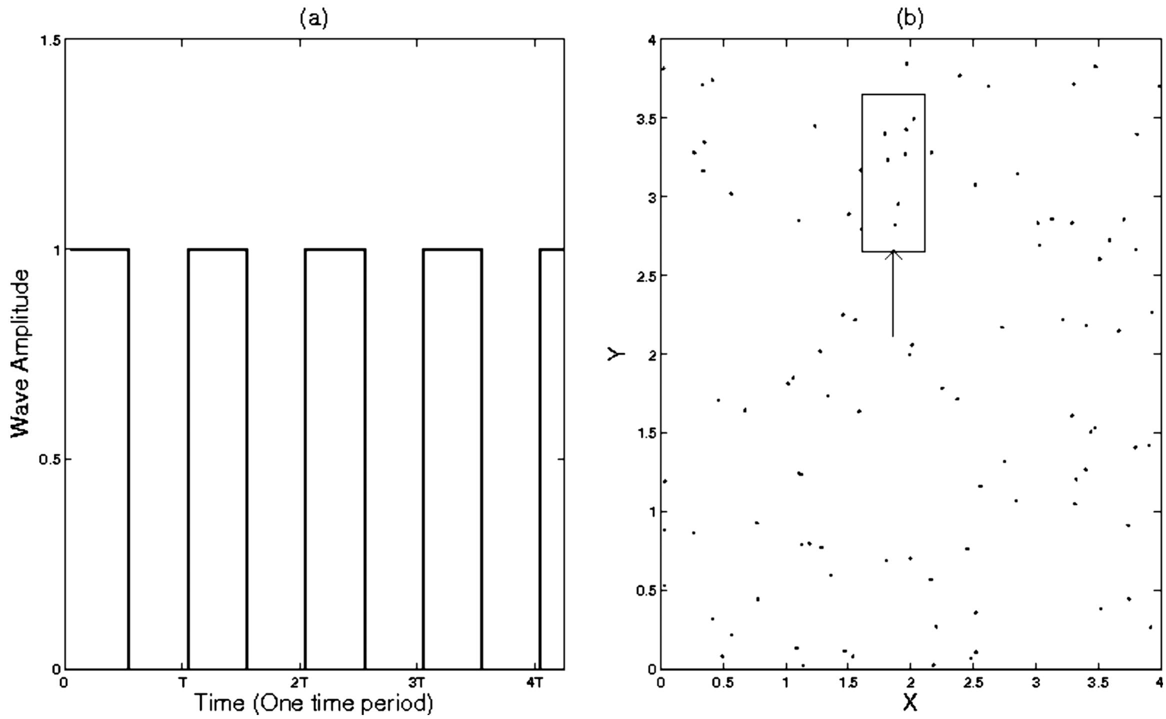

FIG. 2. (a) Typical pulse profile and (b) rectangle of vision for the *i*th oscillator (indicated by the arrow and the large dot): oscillators outside this rectangle are not capable of stimulating this particular oscillator.

linearly with time. The rate of variation of an oscillator's phase is inversely proportional to its time-period. Mathematically stated,

$$\phi_t = \text{modulo}\left\{\left(\frac{2\pi t}{\tau} + \phi_{\text{ini}}\right), 2\pi\right\}, \tag{4}$$

where modulo$\{x,y\}$ means the remainder when x is divided by y, ϕ_{ini} and ϕ_t are the initial phase and phase after a time t (or equivalently after t iterations) of a particular oscillator, and τ is the time-period of that particular oscillator.

2. The protocol

A particular oscillator gets stimulated by a bunch of oscillators which lies within a "rectangle of vision" in front of it. Posterior stimulation is precluded in the current scheme. Figure 2(b) depicts, schematically, the range of vision (R) for the oscillator, i.e., the distance up to which it can get stimulated by a collection of oscillators in front. If (X,Y) are the current coordinates of the ith oscillator, the *rectangle of vision* is defined as the region

$$\left\{(x,y): |x-X| \le \frac{R}{4}, 0 \le y - Y \le R\right\}. \tag{5}$$

Let us assume that there are n oscillators within the rectangle of vision of the ith oscillator. It is to be noted that the rectangular shape is not unique. Any shape which induces a unidirectionality in the system (for example, semicircle) will do the job. For example, earlier, Frasca *et al.*[7] used a circular neighborhood. In the next step, the ith oscillator calculates its phase difference with all the oscillators (n) in the rectangle. This phase difference $\Delta\phi$ is computed as

$$\Delta\phi = \frac{d_i \times |\tau_i - \tau_o|}{\tau_i} + |\phi_i - \phi_o|, \tag{6}$$

where d_i, τ_i, and ϕ_i are the duty cycle, the time-period, and the instantaneous phase of the ith oscillator, respectively, whereas τ_o and ϕ_o ($o = 1 - \text{n}$) are the time-period and instantaneous phase of the oscillator with which the phase difference is being computed. After calculating its phase difference with all the oscillators (n) in the range of vision, the ith oscillator assumes the time-period, duty cycle, and phase of the oscillator with which it has the minimum phase difference. In this sense the coupling is unidirectional and intermediate as mentioned earlier. Subsequently, this ith oscillator performs a spatial movement described later. Thereafter, the same synchronization protocol is followed by the $(i+1)$th oscillator up until the entire array ($i = 1 - N$ which is 100) is exhausted. This procedure is followed recursively until the entire array gets synchronized, i.e., global synchronization is achieved, or up to a predetermined number of iterations, whichever is less.

C. Hopping process

Three possible scenarios of spatial movement are entertained in the present work. First, the oscillators remain static. Second, all the oscillators exhibit harmonic hopping of a predetermined amplitude. Finally, random hopping is considered, in which although the hopping amplitude for all the oscillators is maintained constant, the hopping direction is

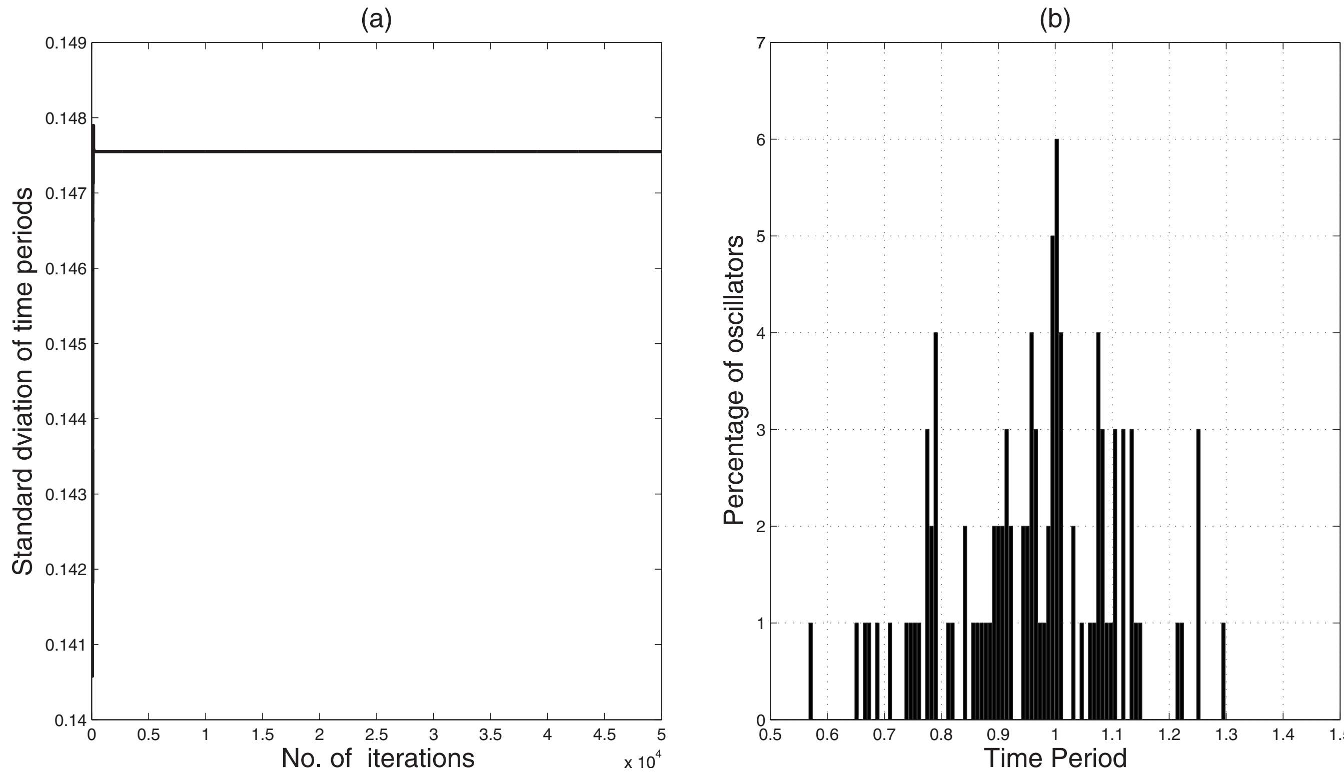

FIG. 3. Asymptotic static case configuration. (a) Standard deviation of time-periods for the ensemble. (b) Distribution of time-periods for the oscillators.

chosen stochastically. For all the above cases, the motion of the oscillators remains confined to a two dimensional plane of predetermined size. This is achieved by making the boundaries of this confining plane completely inelastic, such that if an oscillator tries to hop beyond the boundary it will get stuck to it. In the following iteration it will jump inward and consequently remain confined. However, we also carried out simulations with periodic boundary conditions and the main results of the present work seem to persist. In the following section (Sec. III), we compare and contrast the different synchronization phenomena observed for distinct types of hopping.

D. Algorithm

To summarize, the whole prescription for synchronization is presented below in an algorithmic form.

1. Algorithm

(1) Measure the phase difference with all the oscillators in the rectangle of vision.
(2) Choose the oscillator with which minimum phase difference is measured.
(3) Couple with that oscillator and copy its time-period, duty cycle, and phase.
(4) Hop to a new place. Other oscillators also hop.
(5) Repeat the process sequentially for other oscillators.
(6) Repeat steps 1–5 outlined above until the last (100th) oscillator has been updated.
(7) Steps 1–6 are repeated up to a fixed number of iterations or the number of iterations required for complete synchronization, whichever is less.

III. SYNCHRONIZATION AND HOPPING

Simulations were employed to analyze the correlation between the extent of synchronization and characteristics of the spatial movement, namely, H (hopping amplitude) and θ (hopping direction in body coordinates). The standard deviation of time-periods of the oscillators acts as a measure of synchronization. Global synchronization is manifested by zero standard deviation. To reiterate, the three different cases discussed are the following:

(1) Static case (zero hopping), $H=0$ for all oscillators; θ not used.
(2) Harmonic hopping, $H=$const and identical for all oscillators; θ remains constant in time but is distributed randomly between $[0,2\pi]$ among the oscillators.
(3) Random hopping, $H=$const and identical for all oscillators; θ distributed randomly between $[0,2\pi]$ in space and time.

A. Static case

For the case of zero hopping the oscillators do not exhibit global synchronization. As shown in Fig. 3, the standard deviation of the time-periods for the ensemble settles on a nonzero value. This is due to the fact that the oscillators are not globally coupled and can interact only with a small

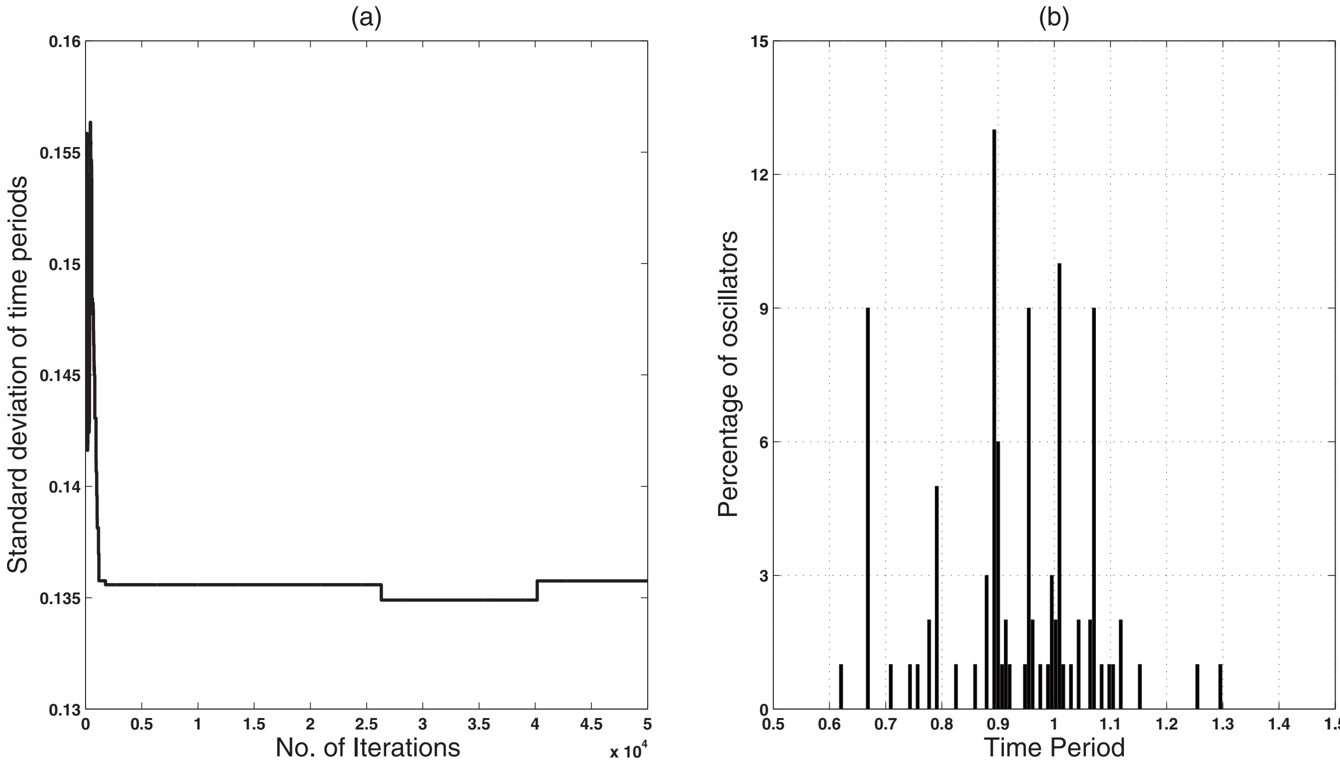

FIG. 4. Asymptotic harmonic hopping configuration. (a) Standard deviation of time-periods for the ensemble. (b) Distribution of time-periods for the oscillators.

fraction of the whole population. Furthermore, since the oscillators are frozen in space, this small fraction remains constant and interaction with new oscillators is forbidden. Now, once the *i*th oscillator updates its time-period and duty cycle with a jth oscillator in its rectangle of vision, its $\Delta\phi$ becomes 0 with respect to the jth oscillator. Since absolute value of the phase difference is used as a measure for all successive iterations, the *i*th oscillator gets stimulated by the same os-

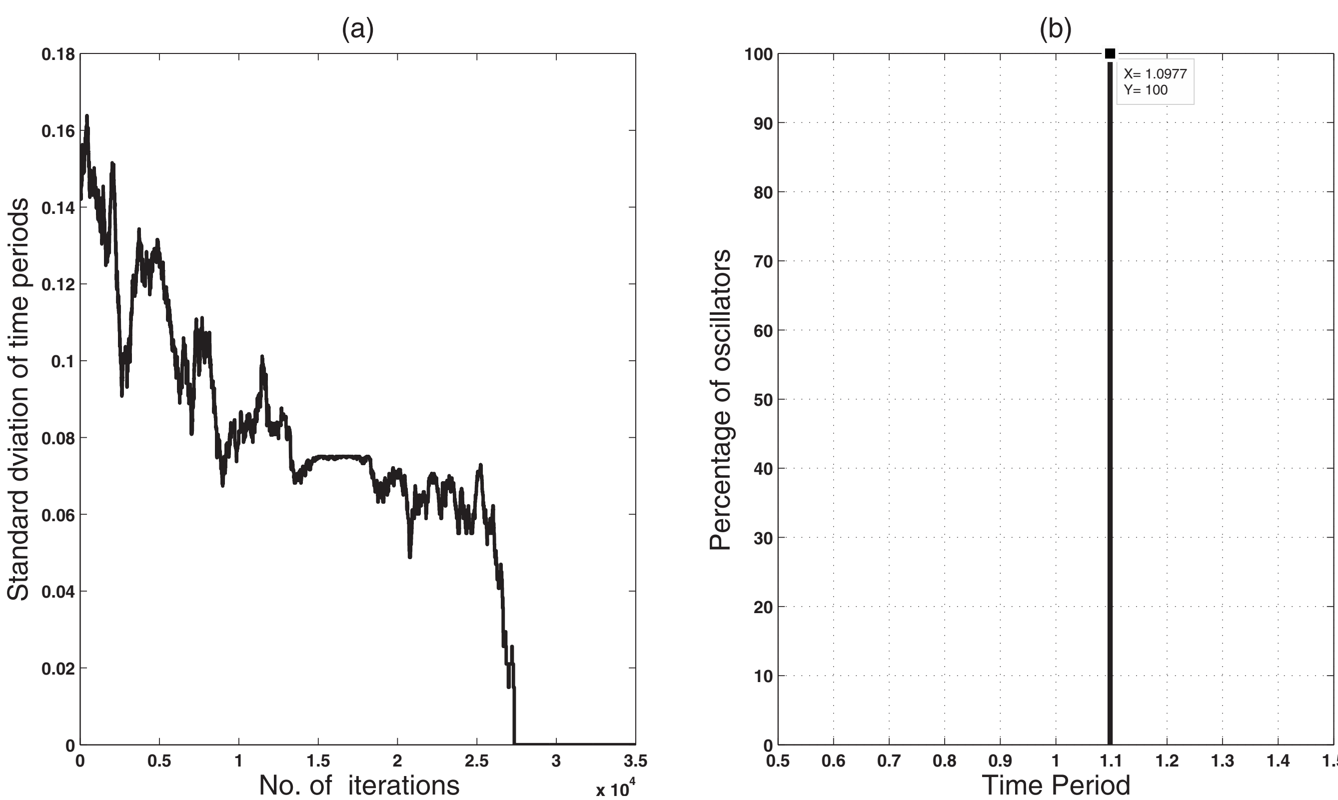

FIG. 5. Asymptotic random hopping configuration. (a) Standard deviation of time-periods for the ensemble. (b) Distribution of time-periods for the oscillators.

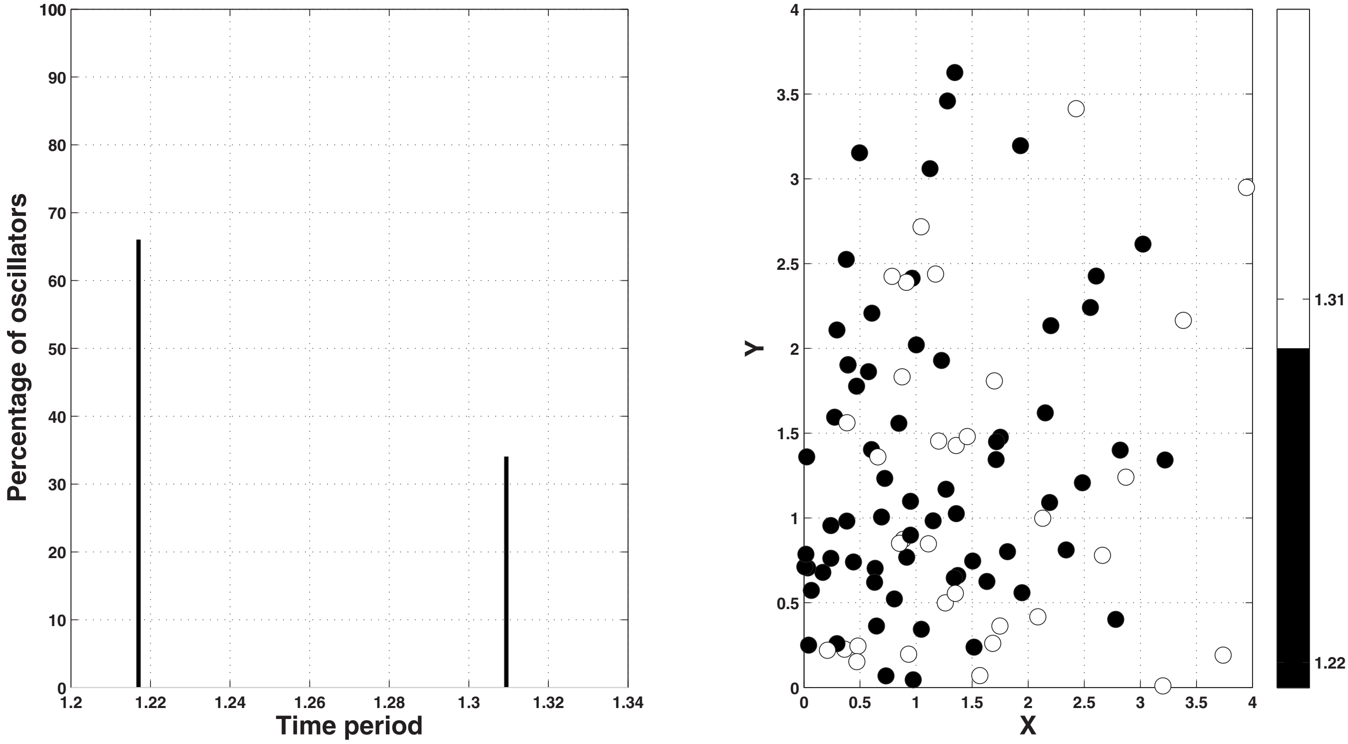

FIG. 6. The left panel shows the emergence of two clusters of oscillators with different time-periods. The right panel depicts the spatial distribution, in the XY plane, of these two groups of oscillators with distinct time-periods.

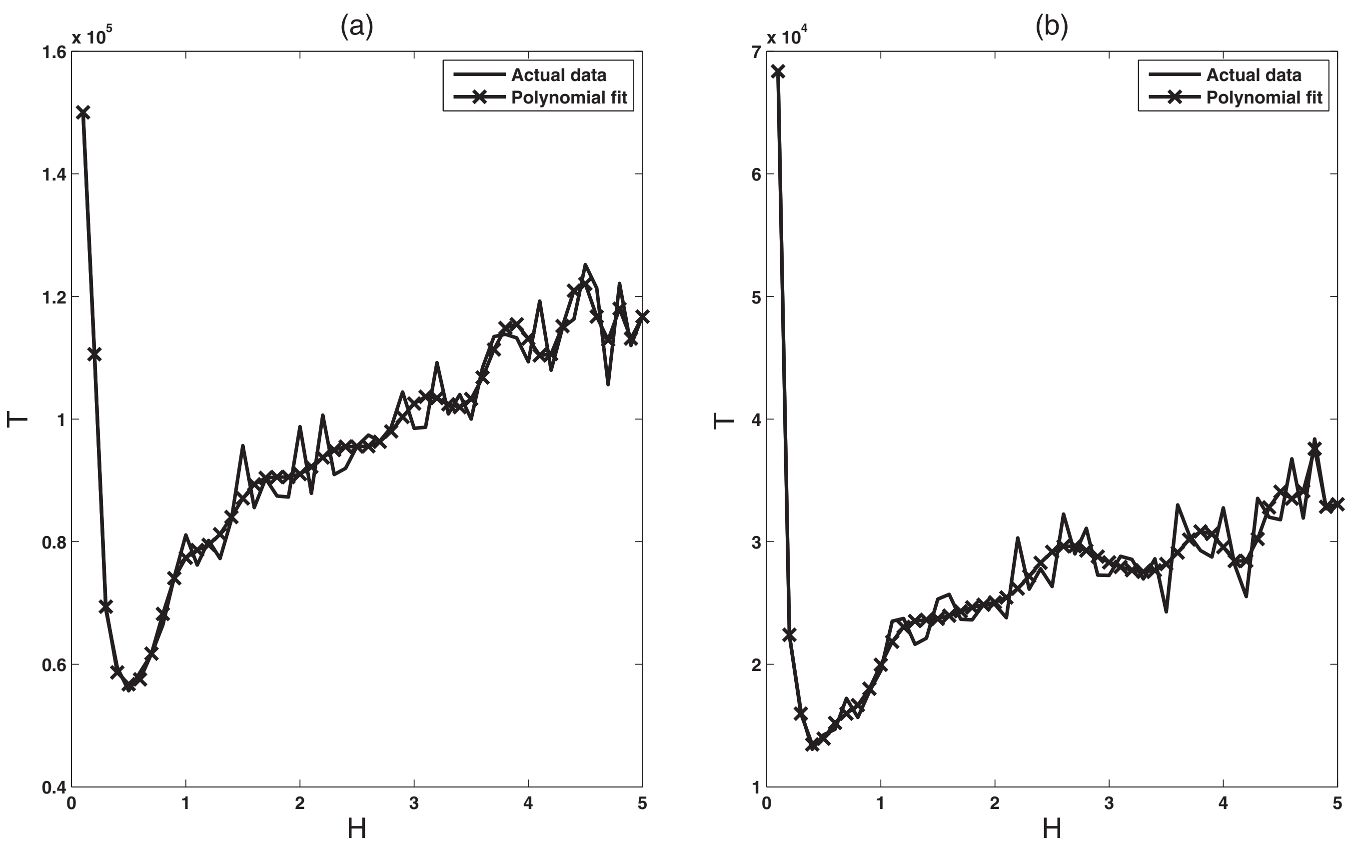

FIG. 7. Time required (No. of iterations) (T) to achieve global synchronization vs hopping amplitude (H) of an ensemble confined to (a) 4×4 and (b) 5×5 container. An average of 50 numerical runs along with a 20th degree polynomial fit is presented.

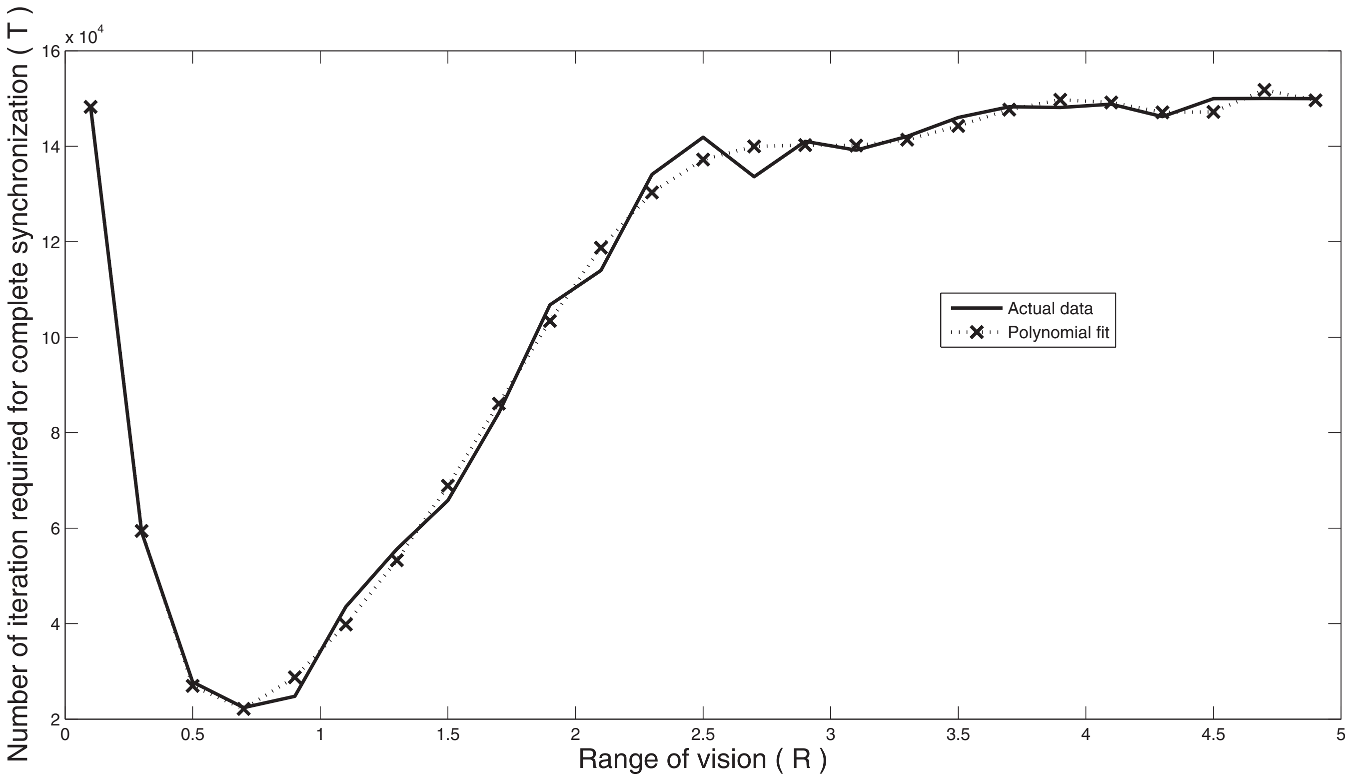

FIG. 8. Time required (No. of iterations) (T) to achieve global synchronization vs range of vision (R) of an ensemble confined to 4×4 container. An average of 20 numerical runs along with a tenth degree polynomial fit is presented.

cillator (jth) each time around rendering the time-period profile stagnant and global synchronization unattainable.

B. Harmonic hopping

In case of harmonic hopping the value of the hopping amplitude H is the same for the entire ensemble. However, the hopping direction (uniquely determined by θ) is distributed randomly, between $[0,2\pi]$, among the hundred oscillators. Both θ and the hopping amplitude are maintained constant for all successive iterations. Numerical results of Fig. 4 indicate that only partial synchronization is achieved for the case of harmonic hopping, the reason being that the harmonic spatial motion of the oscillator does not facilitate interaction of a particular oscillator with the entire population. Analogous to the static case, the ith oscillator gets stimulated by the same set of oscillators, although the number of stimulating oscillators is augmented due to the hopping dynamics. Consequently, the extent of synchronization is increased in comparison to the static counterpart. However, since harmonic spatial motion provides only restricted mixing among the population of oscillators, the standard deviation of time-periods converges at a nonzero value. This nonzero standard deviation [Fig. 4(a)] in conjunction with the distribution of time-periods for the oscillators [Fig. 4(b)] points to the inception of partial synchronization and cluster formation.

C. Random hopping

For random hopping the amplitude H is fixed at the same value for all oscillators. However, the value of θ is distributed randomly in space and time. This implies that θ is assigned randomly between 0 and 2π among the different oscillators in the population. Moreover, θ is updated (a new value chosen randomly between 0 and 2π) for each oscillator at every iteration. Numerical results of Fig. 5 indicate that for the random hopping scheme global synchronization is achieved in the collection of oscillators. The reason being that by virtue of random hopping, there is an enhanced probability that a particular oscillator can interact with a sufficient number of other oscillators enabling the entire ensemble to synchronize. Moreover, the time taken to achieve global synchronization has a nonmonotonic dependence on the hopping amplitude. This effect is studied in the following section (Sec. IV A). To summarize, the zero standard deviation [Fig. 5(a)] in conjunction with the single peak distribution of time-periods for the oscillators [Fig. 5(b)] points to the inception of global synchronization for the case of random hopping. Also, for inappropriate value of hopping amplitude the system often locks in two or three clusters in space and time. One such two cluster spatiotemporal pattern is shown in Fig. 6. Similar clusters are seen for the harmonic hopping case.

IV. DEPENDENCE OF TIME REQUIRED FOR GLOBAL SYNCHRONIZATION ON SEVERAL PARAMETERS

A. Time required for global synchronization versus the hopping amplitude

As seen from Fig. 7, there is a nontrivial dependence of time required for global synchronization (T) on the hopping amplitude (H). Typically, it is observed that T is optimized for some specific value of H. This unimodal profile of T

versus H curve can be considered generic since it is independent of the details (number of oscillators, size of confining plane, etc.) of a particular configuration.

A possible explanation for the observed unimodal profile is that as the hopping amplitude is monotonically increased, mixing of oscillators within the population is augmented. This results in faster attainment of global synchronization. However, if the hopping amplitude is increased even further, the oscillators have a tendency to spend more time in the boundary of the confinement plane, thereby reducing their probability of interaction with the other oscillators of the population. This leads to an increase in the time required to achieve global synchronization for the ensemble. Consequently, there is a range of hopping amplitudes for which the attainment of the global synchronization phenomenon is expedited. The unimodal curve seen in Fig. 7 has the fingerprint of stochastic resonance, an intriguing noise related phenomenon, discussed in numerous systems elsewhere (see Ref. 18 and references therein).

B. Time required for global synchronization versus the range of vision

Interestingly, not only the hopping amplitude but also the range of vision induces a unimodal variation in time required for complete synchronization. This is evident from the results presented in Fig. 8. Therefore, there exists an optimum value of the range of vision as well. From the definition of rectangle of vision it can be inferred that if the range of vision is r, then the size of the rectangle of vision is $r^2/2$. Hence, the size of rectangle of vision is a monotonic function of R. Therefore, the variation in R is equivalent to variation in the size of rectangle of vision. A plausible qualitative explanation for the induction of the unimodal T versus R curve is the following: as we increase the R, for a sufficiently long time the oscillator is coupled to on an average the same set of oscillators, provided the hopping amplitude is not too high. Hence, it is highly likely that the slave oscillator is phase minimized with the same master oscillator. So the updating of the oscillator variables does not take place in quick succession. This results in the higher time required for complete synchronization.

V. CONCLUSIONS

Different features of synchronization for a group of oscillators, subjected to unidirectionally and intermediately coupling, are examined. Our results indicate that the global synchronization of this ensemble is achieved when oscillators execute random spatial movement. As long as the coupling is not rendered global by the choice of range of vision and number density, random hopping ensures global synchronization. We have tested this fact for different system sizes and different boundary conditions. Moreover, the rate of convergence to this globally synchronized state is optimized by the amplitude of random hopping. This is suggestive of a stochastic resonance like behavior, wherein the output of a nonlinear system is optimized via the amplitude of superimposed stochastic fluctuations. These results are of relevance to collective dynamics of biological systems which communicate through pulses, e.g., fireflies. Furthermore, the present scheme could be used to construct gadgets that require high precision synchronization.

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